Class / Semester / Subject Code : S.Y.B.Sc./ III / USMT303 - Discrete Mathematics

Date: 11.10.2019 Duration:[3 Hours] [Total Marks: 100]

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. Let $\mathbb{N}_n = \{1, 2, \cdots, n\}$ for a positive integer n then, $f : \mathbb{N}_n \to \mathbb{N}_n$ is a permutation if
(a) f is one one but not onto (b) f is one one and onto
(c) f is onto but not one one (d) f is any function.

ii. If $\sigma \circ \tau^{-1}$ is an odd permutation then,
(a) Both σ, τ are odd. (b) Both σ, τ are even.
(c) only if σ is odd and τ is even (d) one of the σ, τ is odd and other is even.

iii. Fibonacci Numbers with $f_0 = 1, f_1 = 1$ fibonacci recurrence relation $f_n = f_{n-1} + f_{n-2}$ is linear of
(a) degree one and homogeneous (b) degree two and non-homogeneous
(c) degree two and homogeneous (d) None of these

iv. The characteristic polynomial corresponding to the recurrence $h_n = -25h_{n-1} + 54h_{n-2}$ is
(a) $25x^2 - 54x + 1$ (b) $-25x^2 + 54x$
(c) $x^2 + 25x - 54$ (d) $x^2 - 25x + 54$

v. If X, Y are finite sets and there is an injective function $f : X \to Y$ then
(a) $|X| = |Y|$ (b) $|X| \leq |Y|$ (c) $|X| \geq |Y|$ (d) $|X| < |Y|$

vi. If n and k be positive integers with $n \geq k$, then $S(n, k)$ has recurrence formula
(a) $S(n, k) = S(n-1, k-1) + kS(n, k)$
(b) $S(n, k) = S(n-1, k-1) + kS(n-1, k)$
(c) $S(n, k) = S(n-1, k-1) + kS(n, k-1)$
(d) None of these

vii. The number of pigeons are distributed among k pigeonholes, then at least one pigeonhole contains two or more pigeons is
(a) $k+1$ or more (b) k or more (c) $k-1$ or more (d) None of these

viii. How many 10-letter patterns can be formed from the letters of the word BASKETBALL?
(a) $C(10, 10)$ (b) $\dfrac{10!}{2!2!1!1!1!1!2!}$
(c) $\dfrac{10!}{2! + 2! + 1! + 1! + 1! + 1! + 2!}$ (d) None of these

ix. At a party there are n men and n women. In how many ways can the n women choose male partners for the dance?
(a) D_n (b) $n!$ (c) $(n-2)!$ (d) None of these

x. If $n \geq 2$ is an integer whose prime factorization is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where $\alpha_i \geq 1$, $\forall i, 1 \leq i \leq r$, then
(a) $\phi(n) = n\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2}\right)\ldots\left(1 + \frac{1}{p_r}\right)$ (b) $\phi(n) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)$
(c) $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)$ (d) None of these

2. (a) Attempt any **ONE** question from the following: (8)

   i. If the pair of cycles $\alpha = (a_1, a_2, \ldots, a_m)$ and $\beta = (b_1, b_2, \ldots, b_n)$ have no entries in common, then show that $\alpha\beta = \beta\alpha$.

   ii. Define linear homogeneous recurrence relation. Let q be a non-zero number. Show that $h_n = q^n$ is a solution of the linear homogeneous recurrence relation

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} = 0, (a_2 \neq 0, n \geq 0)$$

   with constant coefficients if and only if q is a root of the polynomial equation $x^2 - a_1 x - a_2 = 0$. Hence prove that if the polynomial equation has 2 distinct roots q_1, q_2 then $h_n = c_1 q_1^n + c_2 q_2^n$ is the general solution of $h_n - a_1 h_{n-1} - a_2 h_{n-2} = 0, (a_2 \neq 0, n \geq 0)$.

   (b) Attempt any **TWO** questions from the following: (12)

   i. Let $\alpha = (1325)(143)(25) \in S_5$ Find α^{-1} and express it as a product of disjoint cycles. State whether $\alpha^{-1} \in A_5$.

   ii. Words of length n using only three letters a, b, c are to be transmitted over a communication channel subject to the condition that no word in which two a's appear consecutively is to be transmitted. Give a recurrence relation for the number of words of length n allowed by the communication channel.

   iii. Find $\sigma \circ \tau, \tau \circ \sigma, \tau^2, \sigma^{-1} \circ \tau^2$ for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 5 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \end{pmatrix}$

   iv. Solve the $h_n = 5h_{n-1} - 6h_{n-2}; h_0 = 1, h_1 = 0$ linear homogeneous recurrence relations by using characteristic equation.

3. (a) Attempt any **ONE** question from the following: (8)

   i. State and prove Addition Principle and Multiplication Principle of Counting.

   ii. Let $S = \{a_1, a_2, \ldots, a_{mn+1}\}$ be a sequence of $mn + 1$ real numbers. Then show that either S has a monotonically increasing subsequence with $m + 1$ terms or a strictly decreasing subsequence with $n + 1$ terms.

   (b) Attempt any **TWO** questions from the following: (12)

   i. How many ways are there to pick 2 different cards from a standard 52 card deck such that:
   (a) The first card is an Ace and the second card is not a Queen?
   (b) The first card is a spade and the second card is not a Queen?

   ii. Prove that a subset of a countable set is either finite or countable.

   iii. Define Stirling number $S(n, k)$ of second kind. Prove that $S(n, n-1) = \binom{n}{2}$

   iv. Show that in any set of six people there are either three mutual friends or three mutual strangers.

4. (a) Attempt any **ONE** question from the following: (8)

   i. Let n be a non-negative integer. Show that

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_n^{n_r}$$

   where the summation extends over all nonnegative integers $n_1 + n_2 + \cdots + n_r = n$.

   ii. State and prove Inclusion-Exclusion principle.

   (b) Attempt any **TWO** questions from the following: (12)

   i. What is the number of ways to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, and u occur consecutively?

   ii. Define permutation on multiset. Consider the multiset $\{3.a, 2.b, 4.c\}$ of 9 objects of 3 types. Find the number of 8-permutations of S.

   iii. The number of circular r-permutations of a set of n elements is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r * (n-r)!}$$

   iv. Define derangement.Write formula for derangement D_n and hence find D_5.

5. Attempt any **FOUR** questions from the following: (20)

   (a) Define an even permutation. Express $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \in S_8$ as a product of disjoint cycles. Determine whether σ is odd or even.

   (b) Find the recurrence relation and give initial conditions for the number of binary strings of length n, that do not have two consecutive $0'$s

   (c) How many two-digit numbers have distinct and non-zero digits?

   (d) State recurrence formula for Stirling Number of Second kind $S(n,k)$ and hence find $S(5,3)$ by using the recursion formula for $S(n,k)$.

   (e) Prove that

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

   (f) Suppose there are 100 students in a school and there are 40 students taking each language, French, Latin, and German. Twenty students are taking only French, 20 only Latin, and 15 only German. In addition, 10 students are taking French and Latin. How many students are taking all three languages? No language?

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