Class / Semester / Subject Code : S.Y.B.Sc./ III / USMT302 - Algebra-III

Date: 09.10.2019 Duration:[3 Hours] [Total Marks: 100]

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x, y, z) = (0, y, x)$, then rank of T^2 is
(a) 0 (b) 1 (c) 2 (d) 3

ii. The value of $\text{Det}(e_2, e_1 + 3e_2, -e_3)$, where $\{e_1, e_2, e_3\}$ are columns of 3×3 identity matrix is ...
(a) -1 (b) 1 (c) 0 (d) 3

iii. Which of the following is not a linear transformation from $T : \mathbb{R}^2 \to \mathbb{R}^2$?
(a) $T(x, y) = (x, x + y)$ (b) $T(x, y) = (x + 1, y)$
(c) $T(x, y) = (y, y - x)$ (d) $T(x, y) = (x + y, y - x)$

iv. If A is matrix obtained by replacing k^{th} column of an $n \times n$ identity matrix by X^t where $X = (x_1, x_2, \ldots, x_n)$ then $\det A = \ldots$
(a) $x_1 + x_2 + \cdots + x_n$ (b) 1 (c) x_k (d) $x_1 x_2 \ldots x_n$

v. If A is 4×7 matrix then
(a) Row space is subspace of $\mathbb{R}^4$ (b) Column space is subspace of $\mathbb{R}^7$
(c) Row space is subspace of $\mathbb{R}^7$ (d) None of these

vi. If $A = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$ then $\det A = \ldots$
(a) $(a - b)(b - c)(a - c)$ (b) $(b - a)(a - c)(c - b)$
(c) $(b - a)(c - a)(c - b)$ (d) None of these

vii. Let $T : V \to V$ be a linear transformation such that $T^2 = 0$ but $T \neq 0$, then
(a) I + T is both injective and surjective
(b) I + T is injective but not surjective
(c) I + T is surjective but not injective
(d) I + T is neither injective nor surjective

viii. Let $S = \{(1, -1), (1, 1)\}$ with usual dot product. Consider the statements:
(i) S is linearly independent set
(ii) S is orthogonal set
(iii) S is orthonormal set, then
(a) Only (i) is True (b) (i) and (ii) are True
(c) (i), (ii) and (iii) are all True (d) None of (i), (ii) and (iii) is True

ix. Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis of inner product space V and $x = \sum_{i=1}^{n} x_i v_i,\ y = \sum_{i=1}^{n} y_i v_i$ be any vectors in V then $< x, y >= \ldots$

(a) $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ (b) $x_1^2y_1^2 + x_2^2y_2^2 + \cdots + x_n^2y_n^2$

(c) $|x_1||y_1| + |x_2||y_2| + \cdots + |x_n||y_n|$ (d) None of these

x. If $\{a, b\}$ is orthonormal basis of $\mathbb{R}^2$ and $x = 3a + 7b,\ y = 3a - 7b$ then $< x, y >= \ldots$

(a) 40 (b) 1 (c) 0 (d) -40

2. (a) Attempt any **ONE** question from the following: (8)

i. Define Linear isomorphism and prove that any $n-$dimensional real vector space is isomorphic to $\mathbb{R}^n$.

ii. Prove that if $L_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation defined by $L_A(X) = AX; A \in M_{m\times n}(\mathbb{R}), X \in \mathbb{R}^n$ then dimension of solution space of $AX = 0$ is $n -$ rank A and hence find dimension of solution space of $\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$.

(b) Attempt any **TWO** questions from the following: (12)

i. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x + y - z, 3x + 4y + 4z, x + y)$ then verify Rank-Nullity theorem for T.

ii. Verify whether the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x, x - y, 2x + y + z)$ is a linear isomorphism or not. If it is invertible then find its inverse.

iii. Examine the consistency and if possible solve the following system.
$2y + 3z = 7, 3x + 6y - 12z = -3, 5x - 2y + 2z = -7$.

iv. Find the solution space and its dimension for the system:
$2x - 3y + z = 0, x + y - z = 0, 3x + 4y = 0, 5x + y + z = 0$.

3. (a) Attempt any **ONE** question from the following: (8)

i. Prove that there is unique $n-$linear skew symmetric function $f : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ with the property that $f(e_1, e_2, \ldots, e_n) = 1$, where $\{e_1, e_2, \ldots, e_n\}$ is standard basis of $\mathbb{R}^n$.

ii. Let $A, B \in M_n(\mathbb{R})$ then show that
(p) $\det(AB) = \det A \det B$ (q) $\det A^t = \det A$.

(b) Attempt any **TWO** questions from the following: (12)

i. Define adjoint of a matrix. Find inverse of $A = \begin{pmatrix}7 & 6 & 5\\ 1 & 2 & 1\\ 3 & -1 & 1\end{pmatrix}$ using adjoint.

ii. Define bilinear map. Further check whether the following map is bilinear,
$f : M_2(\mathbb{R}) \to \mathbb{R}$ defined as $f\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} = a_{11}a_{21} + a_{12}a_{22}$.

iii. Using properties of determinants evaluate determinant of $\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$.

iv. State Cramer's rule and hence find the solution of the system
$2x - y + z = 1, \quad x + 3y - 2z = 1, \quad 4x - 3y + z = 0.$

4. (a) Attempt any **ONE** question from the following: (8)

i. State and prove Cauchy-Schwarz inequality in an inner product space $(V, <, >)$ and verify it for $x = (1, -2)$ and $y = (3, -1)$ in $\mathbb{R}^2$ with Euclidean inner product.

ii. Define inner product space. Further let V be a real inner product space and u be an unit vector in V then define projection $P_u(v)$ of v along u and show that $\| v - P_u(v) \| \leq \| v - \alpha u \|$ for each $\alpha \in \mathbb{R}$.

(b) Attempt any **TWO** questions from the following: (12)

i. Define orthonormal basis of an inner product space V. Further prove that if $\{v_i\}_{i=1}^{n}$ is an orthonormal basis of V and $x = \sum_{i=1}^{n} x_i v_i$ then $x_i = < x, v_i >$ for $1 \leq i \leq n$.

ii. State Traingle inequality for norm and check whether $(\sin x, \cos x)$ satsifies traingle inequality with respect to in inner product $< f(x), g(x) > = \int_{-\pi}^{\pi} f(x)g(x)dx$ in the space $\mathcal{C}[-\pi, \pi]$.

iii. Define orthogonal complement of a subspace W of a vector space V and find orthogonal complement of $W = \{(x, x, x) : x \in \mathbb{R}\}$ in $\mathbb{R}^3$.

iv. Using Gram-Schmidt orthogonalization process find the orthonormal set corresponding to $\{(1, 1, 0), (1, 0, 1), (1, 0, 1)\}$.

5. Attempt any **FOUR** questions from the following: (20)

(a) Show that $A = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$ are row equivalent matrices.

(b) Define subspace of a vector spave V and for linear transformation $T : V_1 \to V_2$, prove that $\ker T$ is a subspace of V_1.

(c) Using determinant check whether $\{(1, 2, 3), (1, -6, 1), (7, 3, 1)\}$ is Linearly independent or dependent and hence state whether $A = \begin{pmatrix} 1 & 1 & 7 \\ 2 & -6 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ is invertible or not with proper justification.

(d) Find the volume of parallelepiped spanned by the vectors $(3, 1, -1)$, $(4, 2, 2)$ and $(-1, 2, -3)$.

(e) Check whether $<, >: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as $< (x_1, y_1), (x_2, y_2 > = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + x_2 y_2$ is an inner product on $\mathbb{R}^2$.

(f) Prove that sum of squares of the diagonals of a parallelogram is equal to the sum of squares of its sides.

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