Class / Semester / Subject Code : S.Y.B.Sc./ III / USMT301 - Calculus-III

Date: 15.10.2019 Duration:[3 Hours] [Total Marks: 100]

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. Which of the following statement is true ?
(a) The set $\mathbb{R}^n$ and ϕ are both finite sets. (b) The set $\mathbb{R}^n$ and ϕ are both open sets.
(c) The set $\mathbb{R}^n$ is open but ϕ is not open set. (d) None of these.

ii. While using polar coordinates , if the limit is in terms of θ then the simultaneous limit
(a) is zero (b) does not exist. (c) is unique. (d) None of these.

iii. The Two path test is useful only for
(a) non existence of limit
(b) continuity.
(c) differentiability
(d) None of the above.

iv. Let $f(x,y) = |x| + |y|$ for $(x,y) \in \mathbb{R}^2$. then
(a) $f_x(0,0) = 0$, $f_y(0,0) = 0$
(b) $f_x(0,0) = 1$, $f_y(0,0) = 1$
(c) $f_x(0,0) = 0$ and $f_y(0,0) = 0$ do not exist.
(d) none of these.

v. Let $f(x,y) = 100 - x^2 - y^2$. Then the direction along which the directional derivative of f at $(3,4)$ is 0 ,is
(a) $(-6,-8)$
(b) $(8,-6)$
(c) $(1,1)$
(d) None of these.

vi. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded . Then
(a) f may or may not be continuous at all point
(b) f is continuous at all points
(c) f is diffrentiable at all points.
(d) None of these

vii. The largest directional derivative of $f(x,y) = x^2y^3$ at the point $(2,3)$ occurs in the direction

(a) -i-j (b) i-j (c) i+j (d) None of these.

viii. The directional derivative of $f(x,y) = x^2 + y^2$ at $(-2,-2)$ in the direction of $u = (1,1)$
(a) -8 (b) 8 (c) 0 (d) -5

ix. Let $u(x,y) = x^2 + y^2; x = r + e^s; y = log\ s$ then $\frac{\partial u}{\partial r}$ is
   (a) $r + e^s$
   (b) $2r + 2e^s$
   (c) r
   (d) e^s

x. Let $f(x,y) = x \sin y$. Then
   (a) f has no critical points.
   (b) f has a critical point which is a local maximum.
   (c) f has a critical point which is a local minimum.
   (d) All critical points of f are saddle points.

2. (a) Attempt any **ONE** question from the following: (8)

   i. Let $\mathbf{S} \subseteq \Re^n$ and $f, g\colon \mathbf{S} \to \mathbb{R}$ be continuous functions at $a \in \mathbf{S}$. Let $\alpha \in \mathbb{R}$. Then prove that
   (i) $f - g$ is continuous at a.
   (ii) αf is continuous at a.

   ii. When do you say that a vector valued function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in \mathbb{R}^n$? Show that f is continuous at a if and only if each $f_i\colon \mathbb{R}^n \to \mathbb{R}$ is continuous at a, where $f = (f_1, f_2, ..., f_m)$

   (b) Attempt any **TWO** questions from the following: (12)

   i. State and prove mean value theorem for derivatives of scalar fields.

   ii. Define limit of a sequence in $\mathbb{R}^n$ and prove that a sequence in $\mathbb{R}^n$ can have at the most one limit.

   iii. Show that $f(x,y,z) = \dfrac{x^2 + y^2 - z^2}{x^2 + y^2 + z^2}$ is not continuous at $(0,0,0)$

   iv. Find a sequence of points $(x_n, y_n) \in \mathbb{R}^2$ such that $\lim\limits_{n\to\infty} f(x_n, y_n) = (2,5)$, where $f(x,y) = (x + y, 2x + 3y)$.

3. (a) Attempt any **ONE** question from the following: (8)

   i. Let U be an open subset of $\mathbb{R}^n$ and $f\colon \mathsf{U} \to \mathbb{R}$ be a scalar function such that all its partial derivative at $a \in \mathbb{R}^n$ exist and are continuous. Then show that f is differentiable at a.

   ii. Let U be an open subset of $\mathbb{R}^n$ and $f, g\colon \mathsf{U} \to \mathbb{R}$ are differentiable function on U. Derive the following properties of the gradient:
      i. $\nabla(\alpha f - \beta g) = \alpha \nabla f - \beta \nabla g$ where α and β are real constant.
      ii. $\nabla(fg) = f\nabla g + g\nabla f$

   (b) Attempt any **TWO** questions from the following: (12)

   i. By using definition prove the functions$f(x,y) = x + y$ is differentiable at the point (1,1).

ii. Find the equation of tangent plane and normal line to the surface $x^3 + y^3 + z^3 = 5$ at the point $a = (1, 1, 1)$

iii. State and prove the Euler's theorem for the homogeneous differentiable scalar valued function.

iv. When do you say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$.Show that such a function is necessarily continuous at a.

4. (a) Attempt any **ONE** question from the following: (8)

i. Let U be an open set in $\mathbb{R}^n$ and $f : \mathsf{U} \to \mathbb{R}^m$ be given by $f(x) = (f_1(x), f_2(x), ..., f_m(x))$ for all $x \in \mathsf{U}$. Prove that f is differentiable at $a \in \mathsf{U}$ if and only if each f_i is differentiable at a and $\mathsf{D}f(a)(u) = (\mathsf{D}f_1(a)(u), \mathsf{D}f_2(a)(u), ..., \mathsf{D}f_m(a)(u))$.

ii. State and prove Taylor's Theorem for a real valued function of two variables.

(b) Attempt any **TWO** questions from the following: (12)

i. For a function $f : \mathbb{R}^n \to \mathbb{R}$ define
(a) relative maximum and minimum
(b) absolute maximum and minimum
(c) critical point (d) stationary point

ii. Find the distance of the plane $5x + 6y + 8z = 10$ from the origin.

iii. Write the Jacobian matrix matrix $\mathbf{J}f(x, y, z, w)$ at $(3, 1, 0, -1)$ of the function $f(x, y, z, w) = (x^2 + 2y, x + z^2 + 3w^2)$.

iv. Let $w = z \tan^{-1} \dfrac{x}{y}, x, y, z \in \Re, y \neq 0$. Find $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$.

5. Attempt any **FOUR** questions from the following: (20)

(a) Compute the directional derivative of $f(x, y) = 6 - 3x^2 - y^2$ at the point $(1, 2)$ in the direction of vector $(1, \dfrac{1}{\sqrt{2}})$

(b) Show that the directional derivatives of $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{Otherwise} \end{cases}$ at the origin exist in the x and y directions but in no other directions.

(c) Find the gradient of function $f(x, y) = e^x \cos y$ at the point $(1, 0)$.

(d) Find the level curve of the function $f(x, y, z) = x^2 + y^2 + z^2$ with given $k = 1, 8, 9$.

(e) Locate and classify the critical points of the functions $f(x, y) = 3x^2 - y^2 + x^3$.

(f) Expand $\sin xy$ around $(2, \dfrac{\pi}{4})$ upto degree two terms and write the remainder term R_3.

****