Class / Semester / Subject Code : F.Y.B.Sc./ I / USMT102 - Algebra-I

Date: 25.11.2019 Duration:[3 Hours] [Total Marks: 100]

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. If $n = 7^3 \cdot 5^4 \cdot 3^5$,$m = 105 \cdot 10^5$ then $g.c.d.(m, n)$ is
(a) 2625 (b) 2645 (c) 1 (d) None of these

ii. If $g.c.d.(a, b) = l.c.m.(a, b)$ then the following is true
(a) $a > b$ (b) $a = b$ (c) $a + b = 1$ (d) None of these

iii. If $a|b$ and $a|c$ then
(a) $a|bx + cy$ (b) $a = 1$ (c) $a|1$ (d) None of these

iv. An integer greater than 1 that is not a prime is termed
(a) even number (b) odd number (c) composite number (d) None of these

v. Let A and B be two non empty sets. Function from A to B is
(a) relation which assigns atleast one element of A to a unique element of B.
(b) relation which assigns every element of A to a unique element of B.
(c) relation which assigns each element of A to a more than one element of B
(d) none of these.

vi. A=$\{1, 2, 3\}$,B=$\{a, b, c, d\}$ then which of the following relation is a function from A to B
(a) $R = \{(1, a), (1, b), (2, c), (3, d), (3, a)\}$
(b) $R = \{(1, a), (2, c)\}$
(c) $R = \{(1, a), (2, b), (3, c), (3, d)\}$
(d) $R = \{(1, a), (2, a), (3, a)\}$

vii. If $*$ is a binary operation on $\mathbb{N}$ then $*$ can be
(a) addition
(b) subtraction
(c) division.
(d) none of these

viii. Consider the binary operation $*$ on $\mathbb{Z}$ as follows
For $a, b \in \mathbb{Z}$,$a * b = a + b - 7$.
The identity of $\mathbb{Z}$ under the binary operation $*$ is
(a) 0 (b) 1 (c) 7 (d) -7

ix. Which is the root of the polynomial $6x^3 - 49x^2 + 51x - 14$
(a) $\frac{-1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{-2}{3}$ (d) -7

x. Degree of constant polynomial is ....
(a) 1 (b) 0 (c) 2 (d) Not defined

2. (a) Attempt any **ONE** question from the following: (8)
   i. State and prove the First Principle of Finite Induction
   ii. Prove that for given integers a and $b(b > 0)$ there exists unique integers q and r such that $a = bq + r, 0 \leq r < b$

   (b) Attempt any **TWO** questions from the following: (12)
   i. Prove the following using second principle of induction
   $x_1 = 1, x_2 = 7, x_{n+1} = 7x_n - 12x_{n-1}, \forall n \geq 2$, then $x_n = 4^n - 3^n$
   ii. For any natural number n prove that the following pairs are relatively prime.
   (p) $2n + 1$ and $9n + 4$, (q) $5n + 2, 7n + 3$
   iii. Prove that $7 | 2222^{5555} + 5555^{2222}$
   iv. If $a \equiv b(mod\ n)$, $c \equiv d(mod\ n)$ then prove that
   (p) $(a + c) \equiv (b + d)(mod\ n)$
   (q) $(a - c) \equiv (b - d)(mod\ n)$
   (r) $ac \equiv bd(mod\ n)$

3. (a) Attempt any **ONE** question from the following: (8)
   i. If $f: X \to Y$, $A \subseteq X$, $B \subseteq Y$ then prove that
   (p)$A \subseteq f^{-1}(f(A))$ and (q) $A = f^{-1}(f(A))$ if and only if f is invective
   ii. If $\sim$ is an equivalence relation on a non empty set X then prove that
   (p)each element of X belongs to some equivalence class of X
   (q)any two equivalence class of X are either disjoint or identical.
   (r)union of these equivalence classes is X

   (b) Attempt any **TWO** questions from the following: (12)
   i. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is bijective.Also find its inverse function.
   ii. Give an example of
   (p) injective function which is not surjective
   (q) surjective function which is not injective
   (r) neither injective nor surjective.
   iii. Show that the relation R defined by for$x, y \in \mathbb{Z}$, xRy iff 2x+3y is divisible by 5 is an equivalence relation on $X = \mathbb{Z}$
   iv. Determine whether the relation $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)(2, 1)(1, 3), (3, 1), (2, 3), (3, 2)\}$ on set $X = \{1, 2, 3, 4\}$ is Reflexive, Symmetric and Transitive and hence an equivalence relation. If R is equivalence relation then find all its equivalence classes.

4. (a) Attempt any **ONE** question from the following: (8)
   i. If $f(x), g(x) \in F[x]$ are non zero polynomials $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ then prove that
   (p) $\deg(f(x) + g(x)) \leq \max\{\deg f(x), \deg g(x)\}$
   (q) $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg g(x)$

ii. If p is a positive prime number then prove that $\sqrt{p}$ is irrational.

(b) Attempt any **TWO** questions from the following: (12)

i. Find G.C.D. of f(x) and g(x) in R[x]:
$f(x) = 2x^3 - 13x^2 + 17x - 3$ and $g(x) = 2x^3 + 5x^2 - 14x + 3$

ii. Find the quotient and remainder when $f(x) = x^4 - 3x^2 + 4x + 8$ is divided by $g(x) = x^2 + 2$

iii. Find the multiplicity of each root of polynomial $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$

iv. Find all the roots of $f(x) = x^3 - 3x^2 - 4x + 12$ if sum of its two root is zero

5. Attempt any **FOUR** questions from the following: (20)

(a) Prove that 1 is the least element of $\mathbb{N}$

(b) State Euler's theorem, Fermat's theorem and Wilson's theorem.

(c) Check whether the following binary operation is commutative and associative.Find an identity element and inverse element if they exist.
$a * b = \frac{ab}{6}$, for $a, b \in \mathbb{Q} - \{0\}$

(d) Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are two functions.Show that for any non empty subset A of X, $g \circ f(A) = g(f(A))$

(e) Define monic polynomial and show that if $f(x)$ is monic polynomial then all rational root of $f(x)$ are integer roots.

(f) Find all fourth roots of unity.

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