Class / Semester / Subject Code : F.Y.B.Sc./ I / USMT101 - Calculus-I

Date:21.11.2019 Duration:[3 Hours] [Total Marks: 100]

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following: (20)

i. The multiplicative identity in $\mathbb{R}$ is
(a) 0 (b) 1 (c) -1 (d) 3

ii. If $\mathbf{A} = [2, 3)$ subset of $\mathbb{R}$ then
(a) $sup\mathbf{A} \in \mathbf{A}$ but $inf\mathbf{A} \notin \mathbf{A}$
(b) $sup\mathbf{A} \notin \mathbf{A}$ but $inf\mathbf{A} \in \mathbf{A}$
(c) $sup\mathbf{A} \in \mathbf{A}$ but $inf\mathbf{A} \in \mathbf{A}$
(d) $sup\mathbf{A} \notin \mathbf{A}$ but $inf\mathbf{A} \notin \mathbf{A}$

iii. Let (x_n) and (y_n) be convergent sequences of real numbers such that $x_n < y_n$, $\forall n \in \mathbb{N}$ then

(a) $\lim\limits_{n\to\infty} x_n < \lim\limits_{n\to\infty} y_n$ (b) $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$
(c) $\lim\limits_{n\to\infty} x_n \geq \lim\limits_{n\to\infty} y_n$ (d) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$

iv. $\lim\limits_{n\to\infty} (1 - \frac{7}{n})^n$ is
(a) e^7 (b) $\frac{7}{e}$ (c) $\frac{1}{e^7}$ (d) does not exist.

v. The graph of $y = \sin x$ intersect the x-axis.....
(a) only at the origin (b) nowhere (c) at infinitely many points (d) None of these.

vi. Select statement from below which is true for each $x \in \mathbb{R}$.
(a) $x < x^2$ (b) $|x^2| = |x|^2$ (c) $|x| < max\{x. -x\}$ (d) $x > 0 \Rightarrow \frac{1}{x} < 0$

vii. If $infB = supB$ then...
(a) B is empty set (b) B is singleton set
(c) B has only two elements (d) None of these.

viii. Let $x_1 = 1$ and $x_{n+1} = \sqrt{14 + 5x_n}$. Assume that the sequences (x_n) converges. Then $\lim\limits_{n\to\infty} x_n$ is
(a) $\sqrt{19}$ (b) -2 (c) 7 (d) 0

ix. Let (x_n) be monotonic increasing sequence which is not bounded above then (x_n)
(a) may be convergent. (b) is convergent. (c) may be divergent. (d) is divergent.

x. Amongst the following ,....... is an odd function.
(a) $y = 17$ (b) $y = 3 \cos x$ (c) $y = x^3 + \sin x$ (d) None of these.

2. (a) Attempt any **ONE** question from the following: (8)
   i. State and prove Cauchy-Schwarz inequality.
   ii. Sate Archimedian property and use it to prove that
   (p) if $x, y \in \mathbb{R}$ such that $x < y + \frac{1}{n}$ $\forall n \in \mathbb{N}$ then $x \leq y$
   (q) if $x \in \mathbb{R}$ then there exists $m, n \in \mathbb{Z}$ such that $m < x < n$.

   (b) Attempt any **TWO** questions from the following: (12)
   i. If $m > 0$ and $x \in \mathbb{R}$ then prove that $|x| \leq m$ if and only if $-m \leq x \leq m$
   ii. If $a, b, c \in \mathbb{R}^+$ such that $abc = 8$ then prove that $a + b + c \geq 6$ and $ab + bc + ac \geq 12$
   iii. If $x, y \in \mathbb{R}$ with $x < y$ then prove that there exists a rational number r such that $x < r < y$.
   iv. Prove that if **S** is a non empty subset of $\mathbb{R}$ which is bounded above then the set of its upper bounds is bounded below.

3. (a) Attempt any **ONE** question from the following: (8)
   i. Prove that every Cauchy sequnce is bounded. Is the convese true ? Justify.
   ii. Prove that if $x_n \to p$ and $y_n \to q$ then $3x_n + 2y_n \to 3p + 2q$ with usual notations.

   (b) Attempt any **TWO** questions from the following: (12)
   i. Define Subsequence of a Sequnce .Prove that subsequence of a convergent sequnce converges to the same limit.
   ii. Prove that $\lim_{n\to\infty} a^n = 0$ for $0 < a < 1$.
   iii. Using definition , prove that $\lim_{n\to\infty} \frac{3n-4}{5n-2} = \frac{3}{5}$.
   iv. Show that (x_n) is monotonic and bounded where $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+1} + ... + \frac{1}{n^2+1}$

4. (a) Attempt any **ONE** question from the following: (8)
   i. Let **I** be an open interval in $\mathbb{R}$ and $f : \mathbf{I} \to \mathbb{R}$. Prove that f is continuous at $p \in \mathbf{I}$ if and only if for each sequence (x_n) in **I** converging to p , $(f(x_n))$ converges to $f(p)$.
   ii. State and prove Sandwich theorem for limits of functions.

   (b) Attempt any **TWO** questions from the following: (12)
   i. Using $\epsilon - \delta$ definition, prove that $\lim_{x\to a} \cos x = \cos a$
   ii. Draw the graph of following functions.
   (p) $f(x) = |x| - 4$,for $-3 \leq x \leq 3$
   (q) $f(x) = \lfloor x \rfloor$,for $-3 \leq x \leq 3$
   iii. Discuss the continuity at $p = 0$ and $p = \frac{\pi}{2}$ for the function $f(x) = \begin{cases} 3e^x - 1 & \text{if } x < 0 \\ \cos x + 1 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 2 + \sin^2 x & \text{if } x \geq \frac{\pi}{2} \end{cases}$
   iv. Let **I** be an open interval in $\mathbb{R}$, $p \in \mathbf{I}$ and $f : \mathbf{I} \to \mathbb{R}$ .If $\lim_{x\to p} f(x) = L$ then prove that $\lim_{x\to p} |f(x)| = |L|$. Is the converse true ? Justify.

5. Attempt any **FOUR** questions from the following: (20)

   (a) For a non empty subset $\mathbb{S}$ of $\mathbb{R}$, prove that infimum of $\mathbb{S}$, if exists , is always unique.

   (b) For $x, y \in \mathbb{R}$ prove that
   (p) if $x < y < 0$ then $y^{-1} < x^{-1} < 0$
   (q) if $x < y$ then $x < \frac{x+y}{2} < y$

   (c) State Sandwich theorem for sequnces and use it to prove that $\lim_{n\to\infty} \frac{\sin n}{n} = 0$

   (d) Discuss the convergence of (x_n) where $x_n = \begin{cases} 3 - \frac{2}{n}, & \text{if } n = odd \\ 4 + \frac{1}{n^2}, & \text{if } n = even \end{cases}$

   (e) Give examples of functions f and g such that
   (p) f and g are not continuous but $f + g$ is continuous.
   (q) f is bounded but not continuous.

   (f) Find $\lim_{x\to\infty} \frac{(2x-7)(x^3+\sqrt{2}x^2-5)}{(x+3)(x^2-\sqrt{6}x)(5x+1)}$.

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